An Introduction to
Digital Signal Processing

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Outline

- Signal Processing
- Advantages of DSP
- Application of DSP
- Types of signal
- Basic sequences and operations
- System
- Convolution
- Fourier Series and Fourier Transform

Signal Processing

- Signal processing is concerned with
  - Representation
  - Transformation and
  - Manipulation
    - of signals and the information they contain.

Digital Signal Processing

Concern with the

- Digital representation of signals and
- Use of digital processors to
  - Analyze
  - Modify or
  - Extract information from signals.

Advantages of DSP

- Guaranteed Accuracy
- Perfect reproducibility
- No drift in performance with temperature
- Greater flexibility
- Superior performance
- Adaptive
- Takes the advantages of semiconductor
  - Smaller size
  - Low cost
  - Low power consumption
  - Higher speed

Application areas of DSP

- Image Processing
- Instrumentation
- Speech/audio
- Military
- Telecommunications
- Biomedical
- Consumer applications

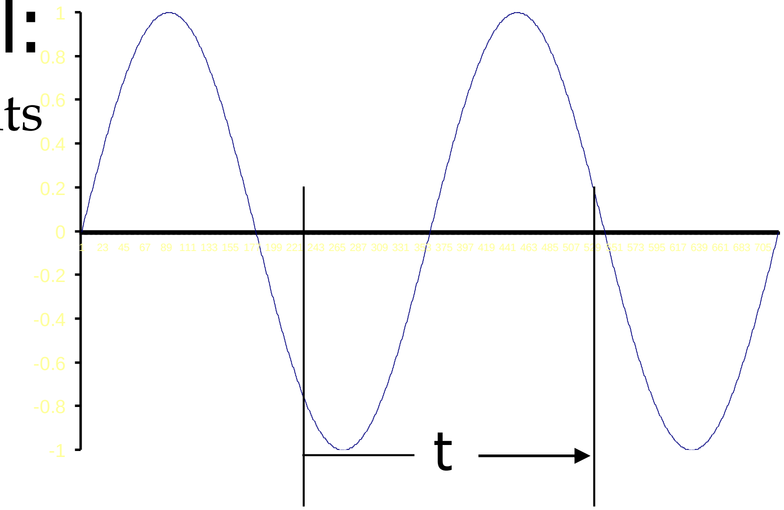
Disadvantages

- Speed and cost
- Design time
- Finite wordlength problem

Types of Signal

- Continuous-time signal:
  - Has infinite number of points between two time interval.
  - Defined along a continuum of times
  - Independent variables are continuous

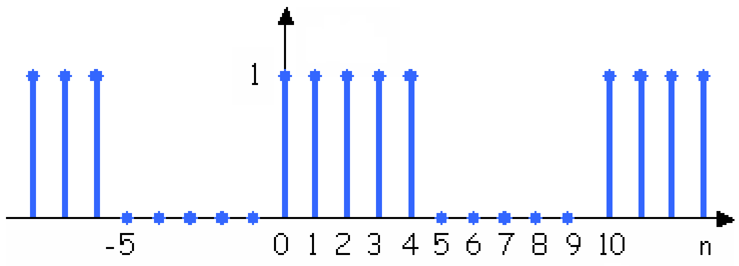
  y=sint



Types of Signal

- Discrete-time signal:
  - Defined at discrete times
  - The independent variables has discrete values
  - Represented as sequence of numbers.

  $y[n]=\{x[n]\}, \ -\infty<n<\infty$

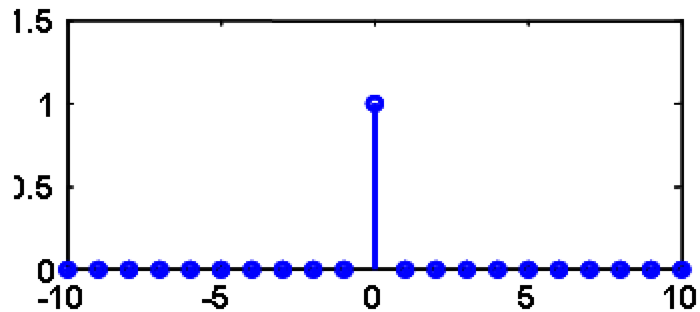


Basic Sequences and Operations

Delaying (Shifting) a sequence

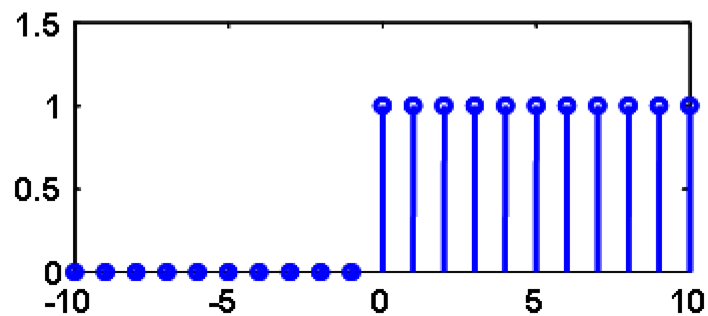
$$y[n] = x[n - n_o]$$

Unit sample (impulse) sequence



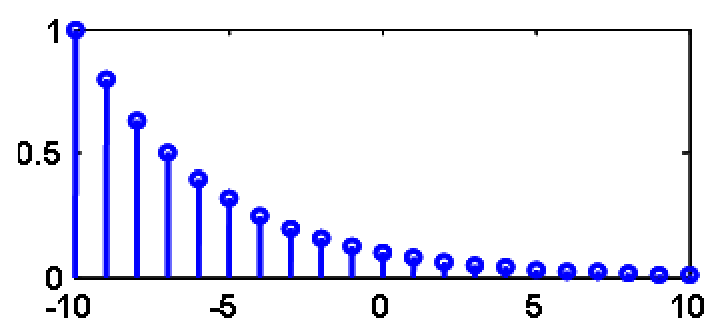
$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Unit step sequence



$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

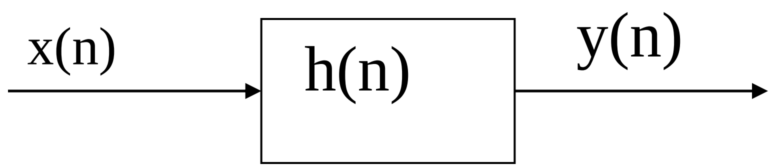
Exponential sequences



$$x[n] = A\alpha^n$$

System

- A system performs an operation on a signal.



X(n) is input, Y(n) is output and
h(n) is impulse response

Example: Filter, amplifier, mixer etc.

Discrete-Time Systems

- Discrete-Time Sequence is a mathematical operation that maps a given input sequence x[n] into an output sequence y[n]

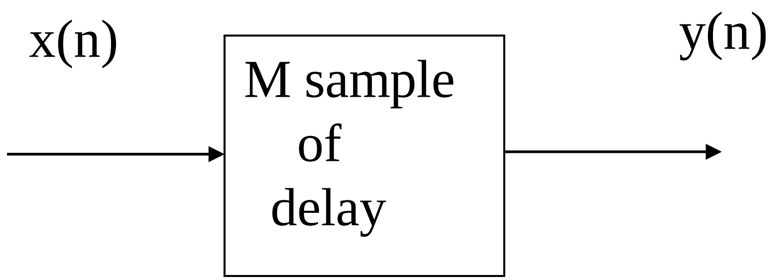
$$x[n] \longrightarrow \boxed{T\{.\}} \longrightarrow y[n]$$

$$y[n] = T\{x[n]\}$$

Example of Discrete-Time Systems

- Ideal Delay System:

  Output is M sample delay of input



$$y[n] = x[n - M]$$

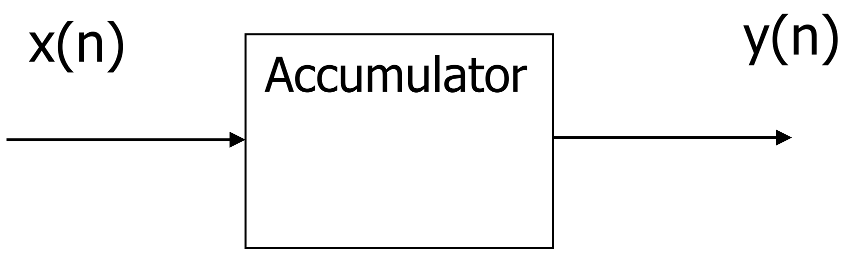
Example of Discrete-Time Systems

- Moving Average System:
  - Computes the nth sample of the output sequence as the average of (M1+M2+1) samples of the input sequence around the nth sample
  - Like a signal moving along a signal

$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$$

Example of Discrete-Time Systems

- Accumulator:
  - Defined as $y(n)=\sum_{n=-\infty}^{n=\infty} x(n)$



If x(n)=δ(n) then y(n)=u(n)

Example of Discrete-Time Systems

- Memoryless System:
  - Output at n depends only on the input at n
  - Don't depend on the past input
- Example Memoryless Systems
  - Square

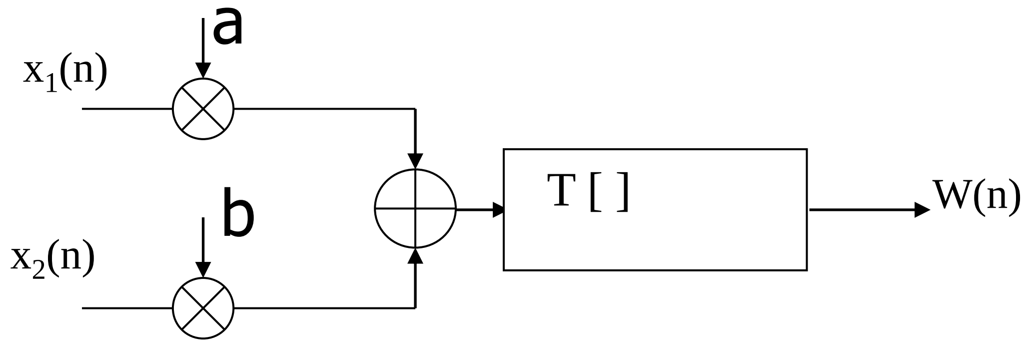
$$y[n] = (x[n])^2$$

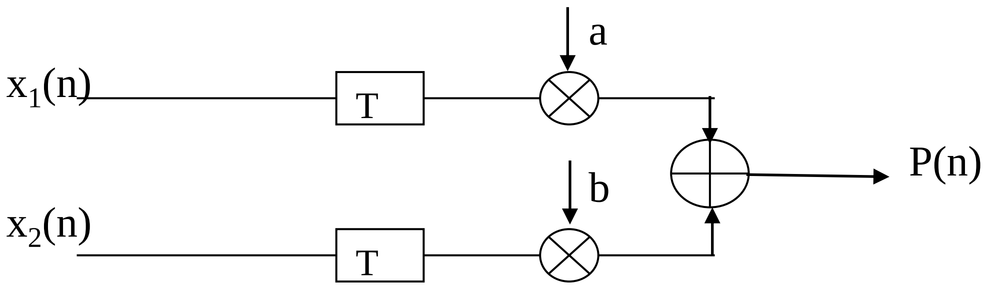
Example of Discrete-Time Systems

- Linear System:

  If the system satisfies principle of superposition

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$$

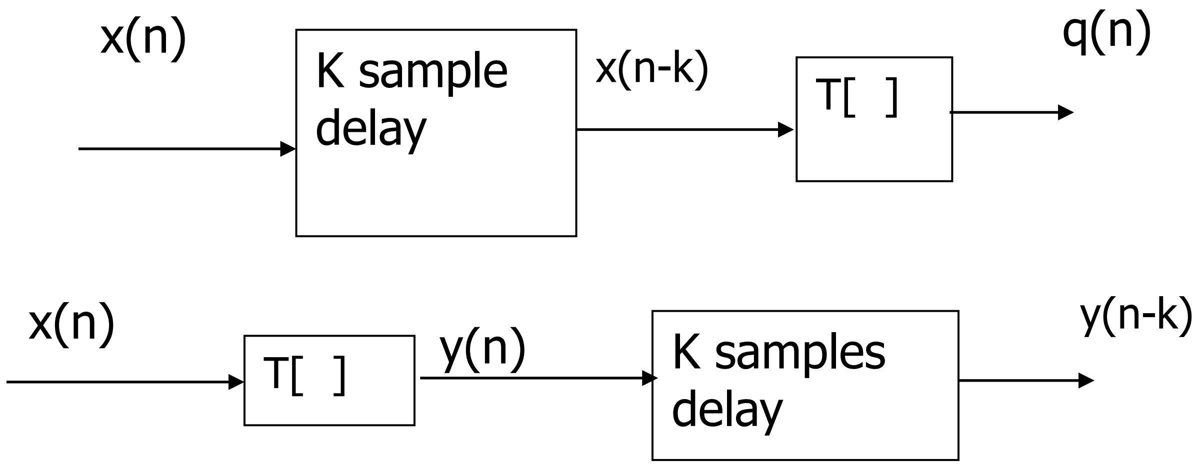




w(n)=p(n)

Example of Discrete-Time Systems

- Shift Invariant System:



Condition: $q(n)=y(n-k)$

$T[x(n-k)]=y(n-k)$.

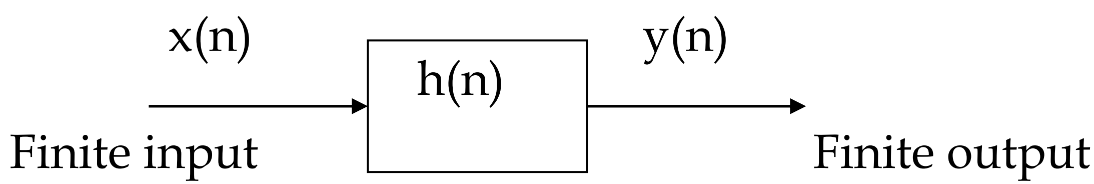
Example of Discrete-Time Systems

- Causality:
  - Output depends on the value of the past history of input

$y(n_0)=f[x(n_0),x(n_{0-1})\ldots\ldots\ldots\ldots\ldots]$

Example of Discrete-Time Systems

- Stability
  - Produces finite output for finite input



- For LSI system $\sum_{n=-\propto}^{\propto} |h(n)| < \propto$

Proof

- For finite input

$$|x(k)| < M$$

$$then \qquad |y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right|$$

$$|y(n)| \le \sum |h(k)||x(n-k)|$$

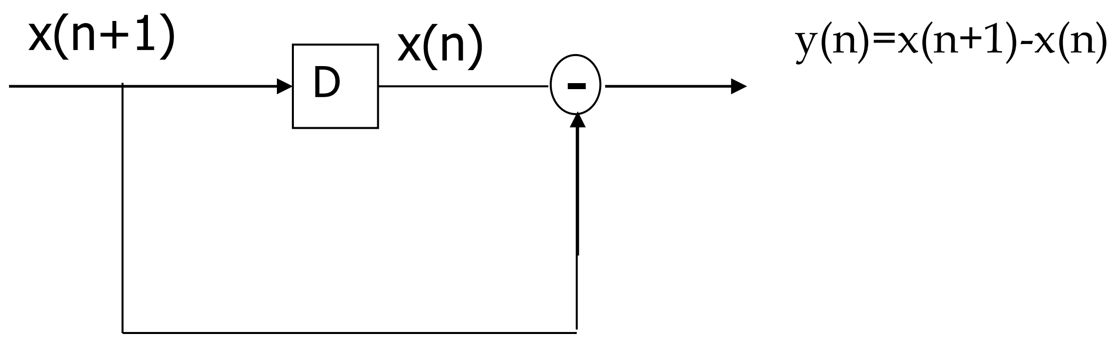
$$|y(n)| < \sum |h(k)|.M$$

$$|y(n)| < M \sum h(k)$$

If $\sum h(k)$ is finite then $|y(n)|$ is finite

Forward Difference System

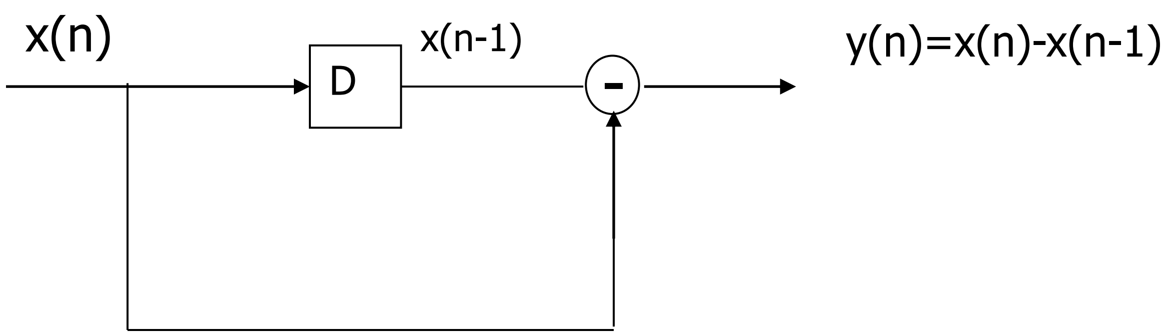
- Subtract a delay version
- It depends on future
- Non-causal system



- Used in lattice structure

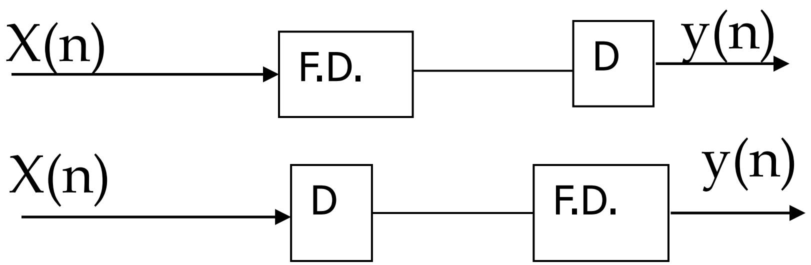
Backward Difference System

- Subtract a delay version
- Depend on past
- Causal system



- Used in DPCM

- Forward difference of one sample delay becomes backward difference



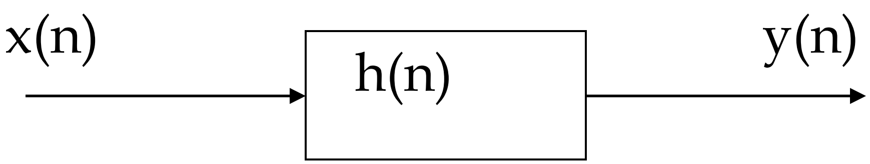
$$h(n)=[\delta(n+1)-\delta(n)]*\delta(n-1)$$
$$h(n)= \delta(n-1) *[\delta(n+1)-\delta(n)]$$
$$h(n)=\delta(n)-\delta(n-1)$$

Convolution

- Convolution of h(t) and f(t) is defined as

$$g(t) = h(t) * f(t) = \int_{\infty}^{\infty} h(\tau) f(t-\tau) d\tau$$

- For discrete system:

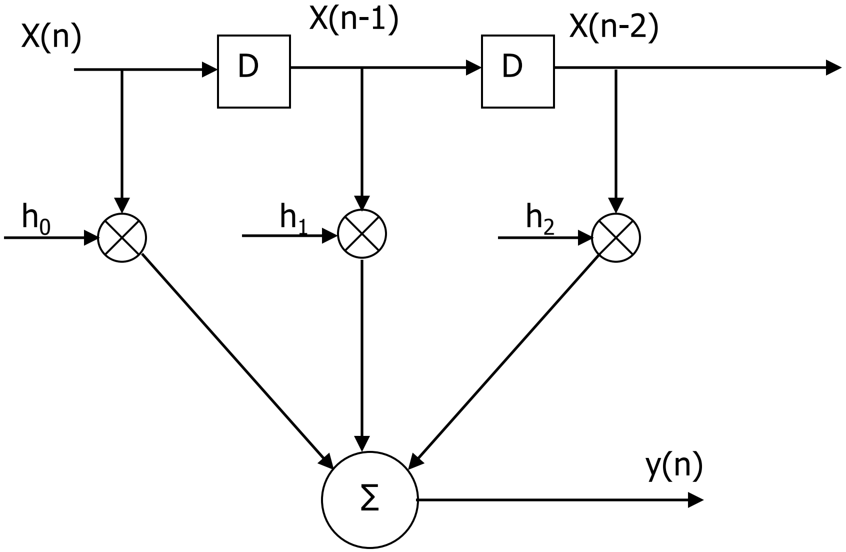


$$y(n) = h(n) * x(n) = \sum_{i=-\infty}^{\infty} h(i) x(n-i) = \sum_{i=-\infty}^{\infty} x(i) h(n-i)$$

- LSI system has the property of convolution

FIR filter

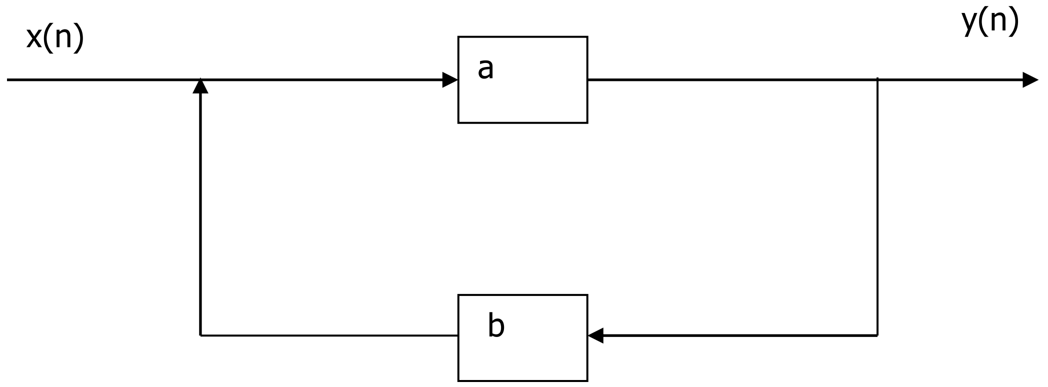
- If h(n) has finite nonzero samples then works as FIR filter



$$y(n)=h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)$$

IIR Filter

- If h(n) has infinite nonzero samples then the system becomes as IIR digital filter



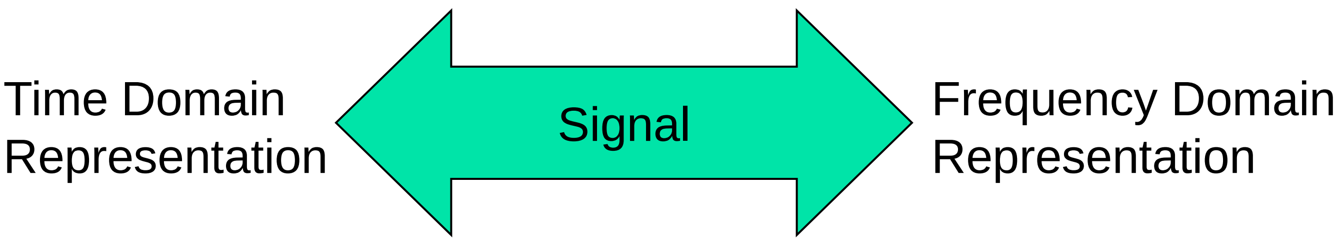
$$y(n)=\sum_{i=0}^{N}a(i)x(n-i)+\sum_{j=1}^{M}b(j)y(n-j)$$

Significance of Convolution

- Convolution in time domain is the multiplication in frequency domain.

Fourier Series

- Signals are functions of time.
- There are two ways by which we can represent the signal.



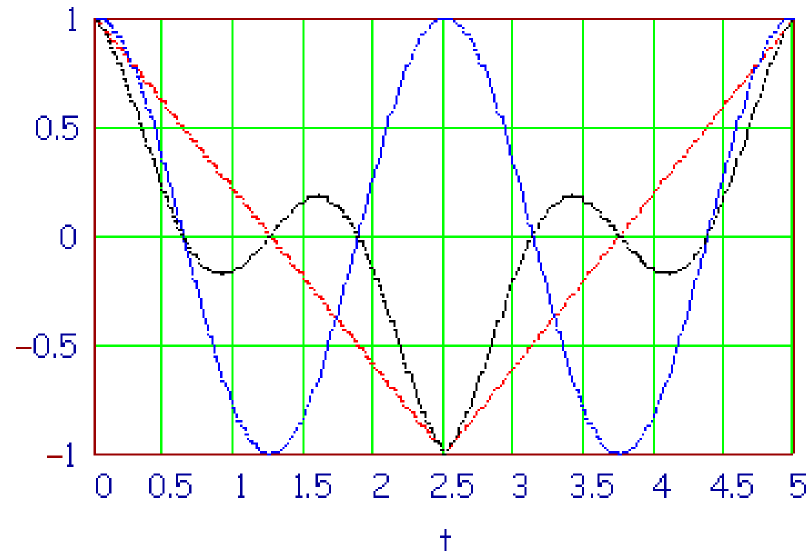
- **Advantages of Frequency response methods**
  - Gives a different kind of insight into a system.
  - It focuses on how signals of different frequencies are represented in a signal. We think in terms of the spectrum of the signal
  - Most of us would rather do algebra than solve differential equations
  - Gives more insight into how to process a signal to remove noise
  - Easier to characterize the frequency content of a noise signal than it is to give a time description of the noise.

Summary

- Be able to compute the frequency components of the signal.
- Be able to predict how the signal will interact with linear systems and circuits using frequency response methods.

The Fourier Series

Fourier, doing heat transfer work, demonstrated that any periodic signal can be viewed as a linear composition of sine waves



"A periodic signal can always be represented as a sum of sinusoids,

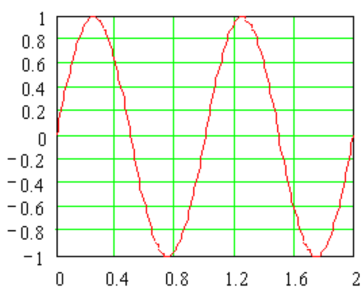
This representation is now called a Fourier Series "

How a signal can be built from a sum of sinusoids?

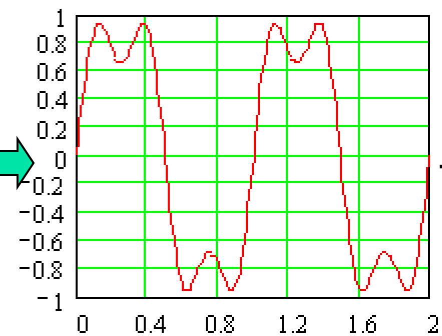
$$Sig(t) = \left[\sum_{i=0}^{\infty} a_i \cdot \cos\left(2 \cdot \pi \cdot i \cdot \frac{t}{T}\right) + b_i \cdot \sin\left(2 \cdot \pi \cdot i \cdot \frac{t}{T}\right)\right]$$

Example:-

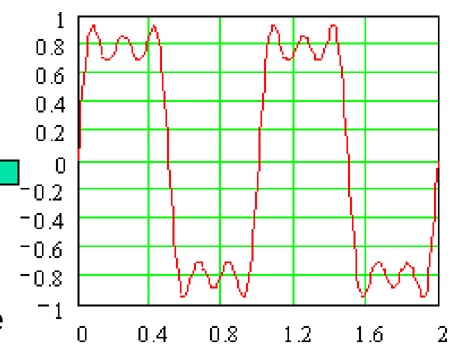
Here is a single sine signal
The expression for this signal is
Sig(t) = 1 * sin(2πt/T)



+ (1/3)sin(6πt/T)

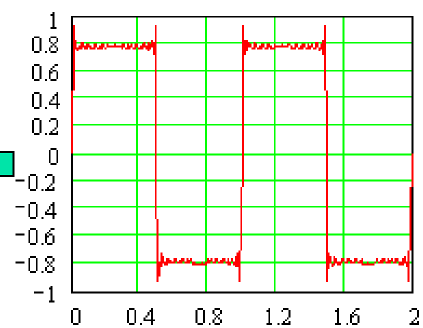


+ (1/5)sin(10πt/T)



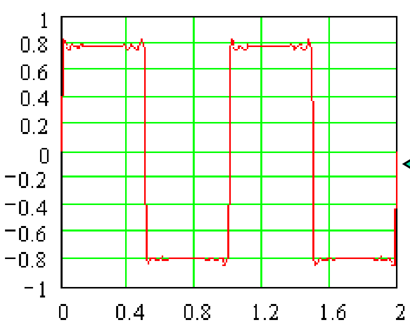
49^{th}

Multiple



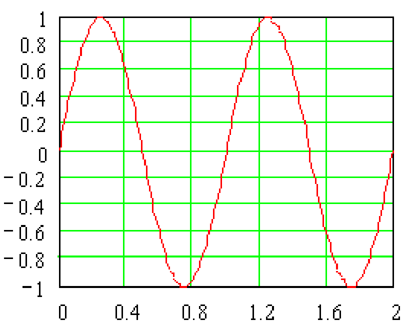
79^{th}

Multiple

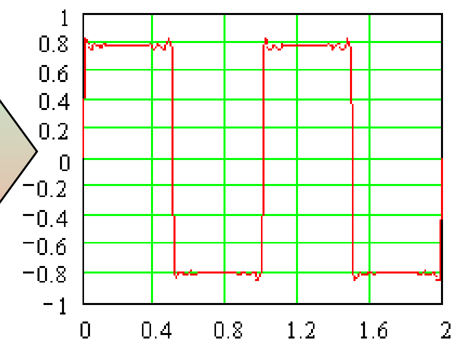


In fact, the way we are building this signal by using Fourier's results.

We know the formula for the series that converges to a square wave. Here's the formula. For a perfectly accurate representation, let N go to infinity.



$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{N} (1/n)\sin(n\omega_0 t)$$



Formulas of Fourier Series expansion:-

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_n = (2/T)\int_0^T f(t)\cos(n\omega_0 t)dt$$

$$b_n = (2/T)\int_0^T f(t)\sin(n\omega_0 t)dt$$

$$a_0 = (1/T)\int_0^T f(t)dt$$

Fourier Transforms

- The *Fourier transform* **(FT) is a generalization of the Fourier series.**

- **Instead of sines and cosines, as in a Fourier series, the Fourier transform uses exponentials and complex numbers.**

- **For a signal or function f(t), the** *Fourier transform* **is defined as**

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Inverse Fourier transform **is defined as**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

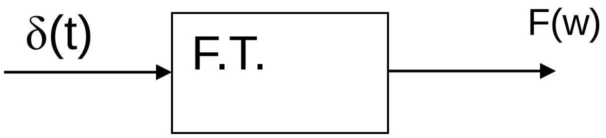
Summary

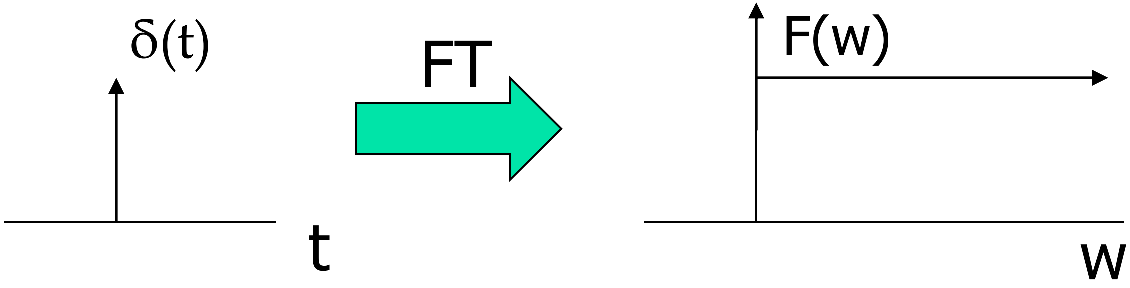
- In general Fourier Analysis can be used to convert the signal domain from time to frequency

- This will help us to interpret the result more quickly and accurately, also changes the system representation to laplace domain.

Significance of Impulse response

FT of Delta function contains all frequency





System can be examined by all frequencies

Frequency Domain Representation Of Discreet Time Signal systems.

General Fourier Equation for frequency representation of sample or signal is:

$$H(e^{jw}) = \sum_{k=-\infty}^{+\infty} h(k) e^{-jwk}$$

Where e is:

$$e^{-jwk} = \cos wk - j \sin wk$$

$$H(e^{jw}) = \frac{\sin(\frac{wn}{2}) e^{j(N-1)\frac{w}{2}}}{\sin(\frac{w}{2})}$$

Frequency Domain Representation Of Discreet Time Signal systems.

The Inverse of frequency representation of Fourier transform

$$h(n) = \frac{1}{2}\int_{-\pi}^{\pi} H(e^{jw})e^{jwn}dw$$

If we know the Impulse response we can use the above equation to design filter using ideal low pass filter where the h(n) is:

$$h(n) = \frac{1}{2}\pi\int_{-w_{co}}^{w_{co}} e^{jwn}dw$$

❖Ideal low pass is not casual and stable

❖It is possible to design filter using low pass. But we need an infinite taps. And if the number of delay tap) is many it is possible to make it sharp and finite.

General definition for even and odd sequence.

Even sequence:

$X_e(n)=X_e(-n)$

Odd sequence:

$X_o(n)=-X_0(-n)$

1. For any sequence in time domain

$$x(n) = x_e(n) + x_o(n)$$
$$x(-n) = x_e(n) - x_o(n)$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$$ Even sequence:

$$x_o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$$ Odd sequence:

General definition for even and odd sequence.

1. For Fourier transform in frequency domain

$$X(e^{jw}) = X_e(e^{jw}) + (e^{jw})$$

Even Part is: $$x_e(e^{jw}) = \frac{1}{2}\left[x(e^{jw}) + x^*(e^{-jw})\right]$$

Odd Part is: $$x_o(e^{jw}) = \frac{1}{2}\left[x(e^{jw}) - x^*(e^{-jw})\right]$$

Even $$X_e(e^{jw}) = X_e{}^*(e^{-jw})$$

Odd $$X_o(e^{jw}) = -X_e{}^*(e^{-jw})$$

Changing jw to -wj is the same because of complex conjugate

Relationship Between Time Domain and Frequency Domain.

where the real part Re of the signal is even part of the sequence

$$F[R_e[x[n]]] = \frac{1}{2}[x(e^{-jw}) + *x(e^{-jw)}) = x_e(e^{jw})]$$

where: R_e =Real part
F =Fourier transform

Transforming Even Part by Fourier Transform is equal to Real part

$$F_1[X_e(n)] = \frac{1}{2}[x(e^{jw}) + x*(e^{jw})]$$

$$= R_e[x(e^{jw})]$$

$$X*(n) \xrightarrow{F_1} X*(e^{-jw})$$

$$X*(-n) \xrightarrow{F_1} X*(e^{jw})$$

The Real part Sequence transformed by Fourier transform equal to even part

$$R_e[x(n)] \quad F_1 \quad X_e(e^{jw})$$

Properties of Fourier Transform In Discrete

Linearity

$A_1x_1(n)+a_2x_2(n) \quad \overset{F_1}{\longleftrightarrow} \quad a_1x_1(e^{jw})+a_2x_2(e^{jw})$

Time Shift and Frequency Shift

$x\,(n-D) = \quad F_1 \quad e^{-jwnD}.x(e^{jw})$ Time Shift

For a shift in time the out put is the multiplication in frequency.

$e^{jwn}.x(n) \quad F_1 \quad X(e^{j(w-w_0)})$ Frequency Shift

Frequency shift is important in communication engineering because shift in frequency does not make any overlap.

Properties of Fourier Transform In Discrete

Time Reversal

$X(-n) \xrightarrow{F_1} X(e^{-jw})$

X(n) is real signal: $X(-n) \xrightarrow{F_1} X^*(e^{jw})$

Differentiation in Frequency Domain

$$n.x(n) \xrightarrow{F_1} \frac{j^{d(x(e^{jw})}}{d\omega}$$

Convolution

$$y(n) = \sum x(k)h(n-k) = x(n) \otimes h(n)$$ Time Domain.

$$y(e^{jw}) = X(e^{jw}).H(e^{jw})$$ Frequency Domain

Properties of Fourier Transform In Discrete

Parseval's Theory

$$E = \sum_{-\infty}^{+\infty} \left| x(n) \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| x(e^{-jw}) \right|^2 d\omega$$

Where E=energy

The Modulation or Windowing Domain

$$y(n) = w(n).x(n)$$

Multiplication in Time Domain but convolution in frequency domain

$$y(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(e^{jw}).w(e^{j(w-\theta)})\ d\theta$$

$h_1 \otimes h_2$ $\quad$ $H_1.H_2$ $\quad$ Discrete convolution

$h_1.h_2$ $\quad$ $H_1 \otimes H_2$ $\quad$ Integration convolution

Chapter- 4
Z-Transform

Defn.

For time sequence x(n) the discrete Z-transform is defined as:

$$x(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

Where Z is complex and can be found any where

- Z is a complex number, but in Fourier transform e^{jw} is represented in unit circle. And it is similar to laplace transform which is analogue.
- Z is infinite but it is possible to make it convergent.

Z-Transform

Example 1

For sequence $x(n)=a^n$ where $n \le 0$ transformed by Z-transform

$$x(z)=\sum_{n=0}^{\infty} a^n z^{-n}$$ Geometric Series

$$x(z)=\sum_{n=0}^{\infty} (az^{-1})^n = 1 + az^{-1} + (az^{-1})^2 + \ldots$$

if $\left|az^{-1}\right| < 1,$ **then** $x[z] = \dfrac{1}{1-az^{-1}}$ **where**

$|z| < |a|$ **The condition for convergence**

Properties of Z-Transform

Linearity Let x(n) and y(n) be any two functions and let X(z) and Y(z) be their respective transforms. Then for any consonants a and b

$$ax(n)+by(n) \quad \xrightarrow{Z} \quad ax(z)+by(z)$$

Shifting

$$x(n+k) \quad \xrightarrow{Z} \quad z^{k}x(z)$$

convolution If w(n)=x(n)*y(n) then

$$w(z)=X(z)Y(z)$$

$$y(n)=\sum h(k)x(n-k) \quad \xrightarrow{Z} \quad y(z)=H(z).X(z)$$

Properties of Z-Transform

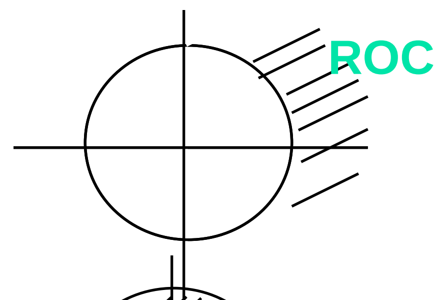
Relationships between Fourier transform and Z-transform

$$x(z) = \sum x(n) z^{-n}$$

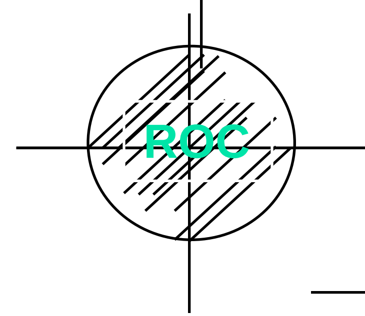
$$x(e^{jw}) = \sum x(n) e^{-jwn} \qquad Z = re^{jw}$$

Range of convergence (ROC) for Z-transform

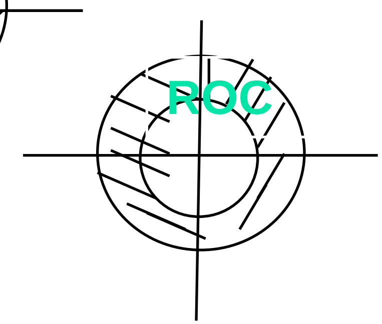
If the sequence is right sided then ROC is:



If the sequence is left sided then ROC is:



If the sequence is both sided then the pole is outside or inside the ring. It is Like donat shape



Inverse of Z-Transform

Method 1

$$x(n) = \frac{1}{2\pi j} \oint_c x(z).z^n \frac{dz}{z}$$

Method 2

$$x(z) = \frac{1}{1 - 0.5z^{-1}}$$

Inverse of Z-Transform

Inspection method

$$a^n a(n) \quad \xrightarrow{Z} \quad \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$-a^n a(-n-1) \quad \xrightarrow{Z} \quad \frac{1}{1-az^{-1}} \qquad |z|<|a|$$

Partial fraction Expansion method

$$x(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad x(z)=\frac{b_0\prod_{k=1}^{M}(1-(kz^{-1})}{a_0\prod_{k=1}^{N}(1-(d_k z^{-1})}$$

$$x(z)=\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}} \qquad A_k=(1-d_k z^{-1})x(z)$$

Inverse of Z-Transform

Power series Expansion

$$x(z) = \sum x(n) z^{-n} = +x(-2) z^{-2} + x(-1) z^{-1} + x(0) + x(1) z^{-1} \ldots$$

$$x(z) = \log(1 + a z^{-1})$$ Which is the result of function or series

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x(n)}{} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x(n) = (-1)^{n+1} \frac{a^n}{n}$$ For $n \geq 1$

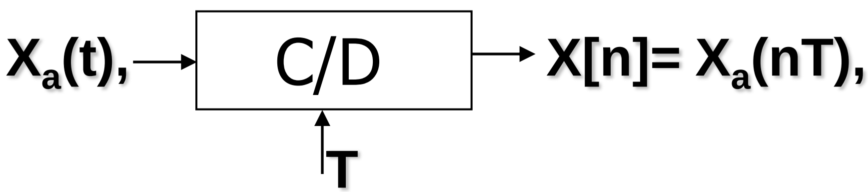
This was the function which is the result of the above function

Chapter 3
Sampling of Continuous-Time Signals

Sampling:

To use digital signal processing methods on analogue signal, it is necessary to represent the signal as a sequence of numbers. this is commonly done by sampling the analogue signal, denoted $X_a(t)$, periodically to produce the sequence

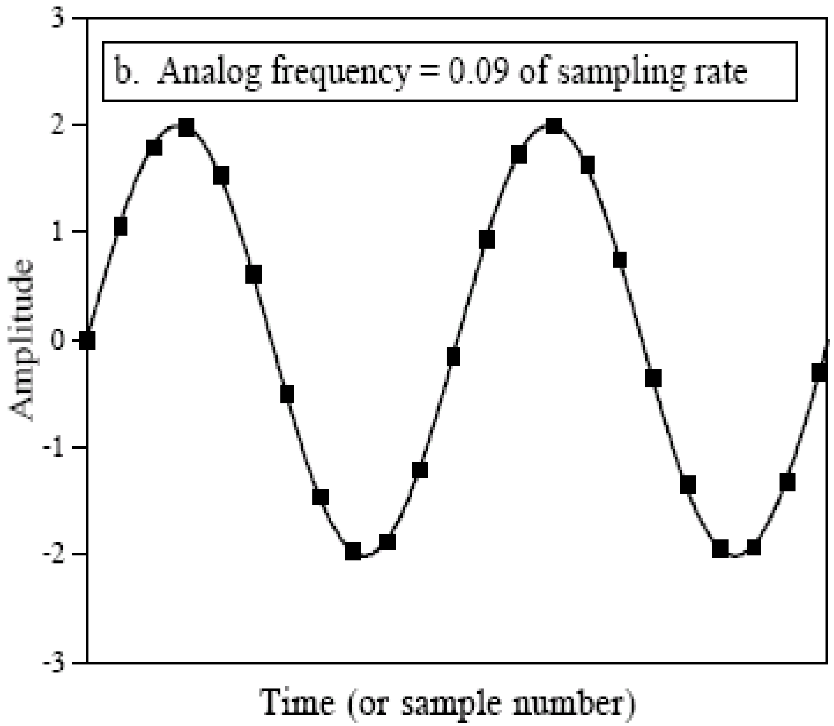
$$X[n]= X_a(nT), \quad -\infty<n<\infty \quad \text{where: } T=\text{sampling period.}$$

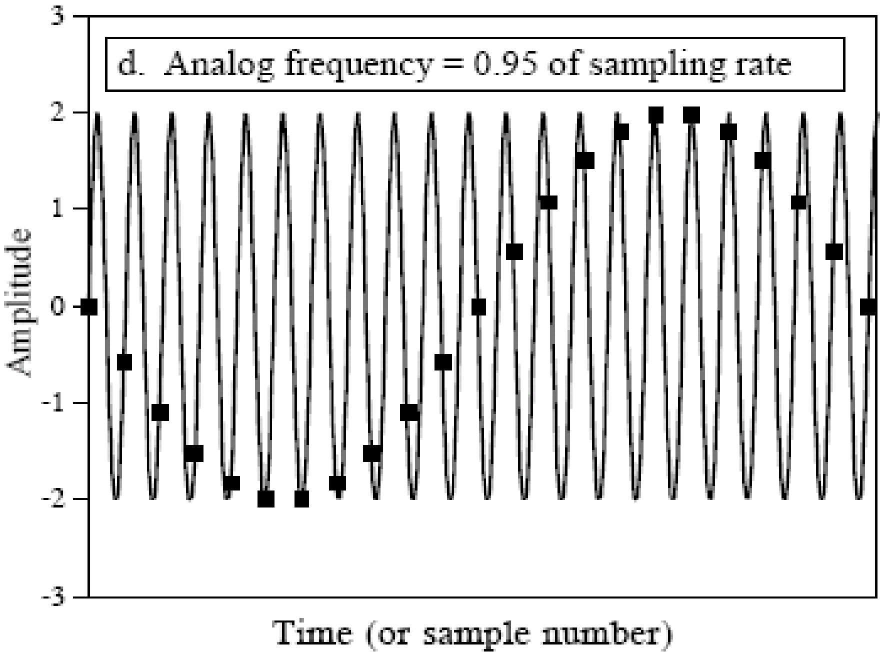


Block diagram representation of an ideal continuous-to-discrete (C/D) converter.

Two examples for sampling

good irreversible





Sampling of Continuous-Time Signals

Fourier transform in analogue :

$$x_a(i\Omega) = \int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}dt$$

Where: Ω =frequency

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_a(j\Omega)e^{j\Omega t}d\Omega$$

$$x(n) = x_a(nT) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_a(j\Omega)e^{j\Omega nT}d\Omega$$

Not periodic

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x\ (e^{j\omega})e^{j\omega n}d\omega$$

periodic (between -Π to Π)

Sampling of Continuous-Time Signals

Sampling Theorem: If a signal $X_a(t)$ has a band limited Fourier transform $X_a(j\ \ )$, such that $X_a(j\Omega)=0$ for $\Omega \geq 2\ \Pi F_N$, then $X_a(t)$ can be uniquely constructed from equally sampled spaces
$X_a(t)$, $-\infty<n<\infty$ if $1/T >2F_N$.

If the Fourier transform of Not periodic $X_a(t)$ is defined as:

$$x_a(i\Omega) = \int_{-\infty}^{+\infty} x_a(t)e^{-j\Omega t}dt$$

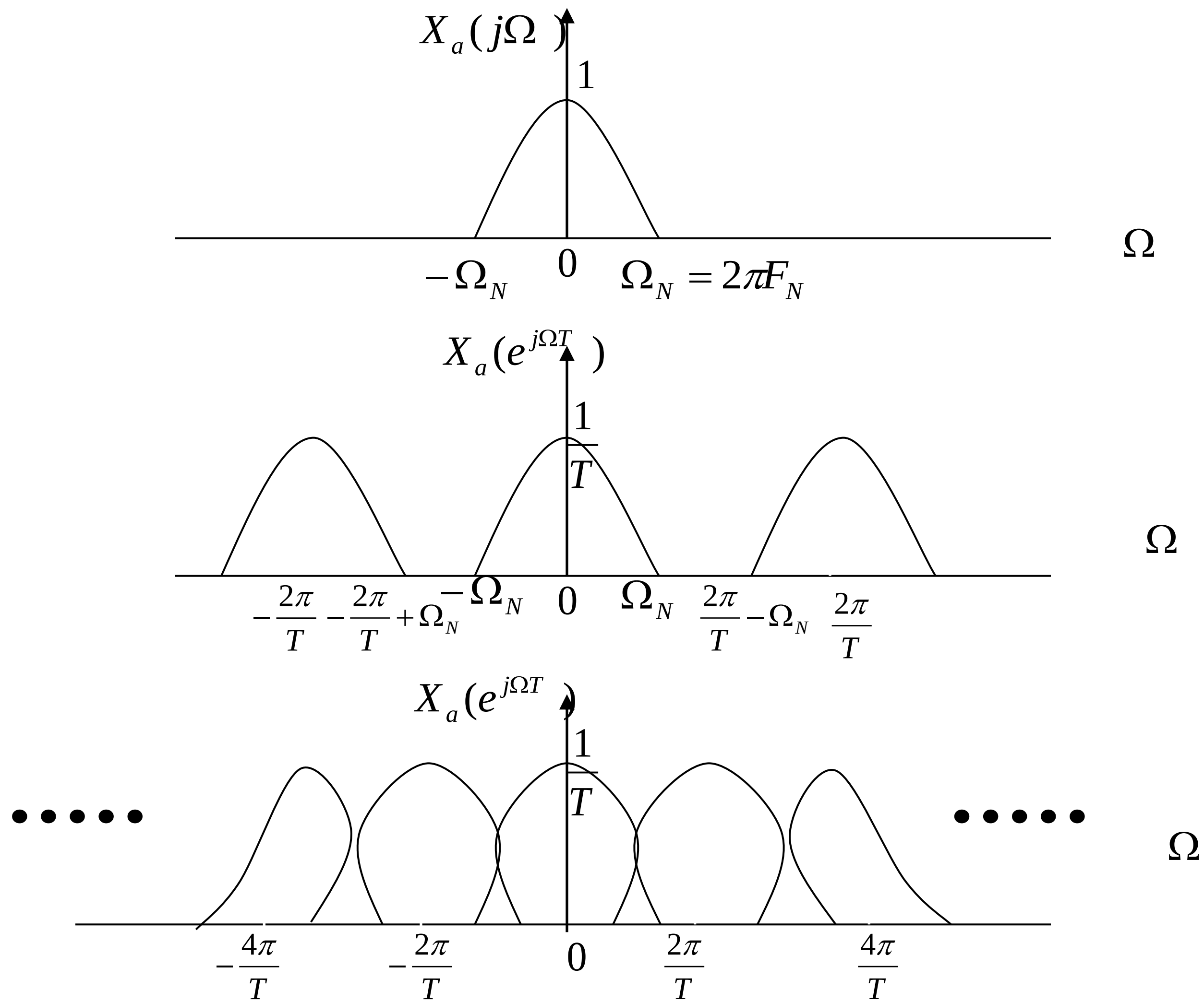
then If $(e^{j\omega})$ is evaluated for frequency $\omega = \Omega T$ then, $X(e^{j\Omega T})$ is related to $X_a(j\Omega\ )$ by:

$$X(e^{j\Omega t}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(j\Omega + j\frac{2\pi}{T}k)$$

where: K=integer

Recovery of Analogue Signal From Sampling (1)

Illustration of Sampling:



Recovery of Analogue Signal From Sampling (2)

In the above figures:

Figure (a) assume that $X_a(j\Omega) = 0\_ \; for \; \_|\Omega| > \Omega_N = 2\pi F_N$ the Frequency F_N is called **Niguist Frequency.**

Figure (b) depicts the case when $1/T>2F_N$ so that the image of transform don't overlap into the base band $|\Omega| < 2\pi F_N$

Figure (c) on the other hand shows the case $1/T>2F_N$. In this case the image centered at 2 Π/T overlaps into the base band. This condition, where a high frequency seemingly takes on the identity of the lower frequency is called aliasing.

Aliasing can be avoided only if the Fourier transform is band limited and The sampling frequency (1/T) is equal to at least twice the Niguist Frequency ($1/T>2F_N$)

Recovery of Analogue Signal From Sampling (3)

Under the condition $1/T > 2F_N$ the Fourier transform of the sequence of sample is proportional to the analogue signal in the base band; i.e.,

$$X(e^{j\Omega T}) = \frac{1}{T} X_a(j\Omega) \qquad |\Omega| < \frac{\pi}{T}$$

Using this result, it can be shown that the original signal can be related to the sequence of samples by Interpolation Formula

$$\boxed{x_a(t) = \sum_{n=-\infty}^{+\infty} x_a(nT) \left[\frac{\sin[\pi(t-nT)/T}{\pi(t-nT)/T} \right]}$$

Interpolation Formula

If the samples of a band limited analogue signal taken at a rate of at least twice the Niguist Frequency, it is possible to reconstruct the orignal analogue signal using the above equation.

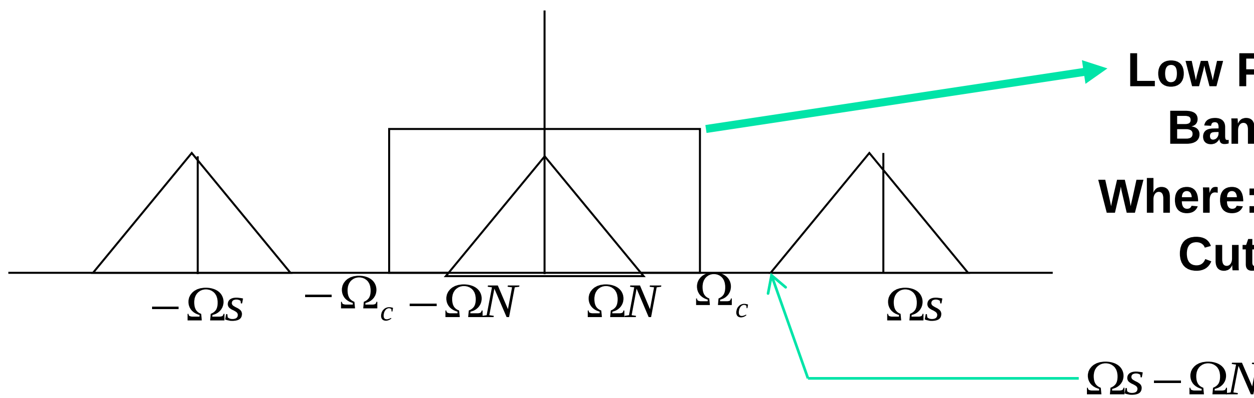
Recovery of Analogue Signal From Sampling (4)

General formula for **Interpolation.**

$$X_a(t) = \sum_{k=-\infty}^{\infty} C_K \phi_k(t)$$

Where: $C_k =$ **Sampling Value**

$\phi_k =$ **Sink**



Low Pass Filter Band width

Where: Ω_c **is Cutoff frequency**

If $\Omega_N < \Omega_c < \Omega_s - \Omega_N$ **then it is possible to recover. If the above condition is not set, then aliasing will be produced.**

Recovery of Analogue Signal From Sampling (5)

order to avoid aliasing (distortion)

$$\boxed{\Omega_s - \Omega_N > \Omega_N}$$

Condition for anti aliasing

$$\boxed{\Omega s > 2\Omega_N}$$

Nyquist Frequency Rate

Since $\Omega_s = \dfrac{2\pi}{T}$ **we can write the above equation as:**

$$\boxed{\Omega_N < \frac{\pi}{T}} \quad \text{or} \quad \boxed{F_N < \frac{F_s}{2}} \quad \text{or} \quad \boxed{\Omega_s > 2\Omega_N}$$

Recovery of Analogue Signal From Sampling (5)

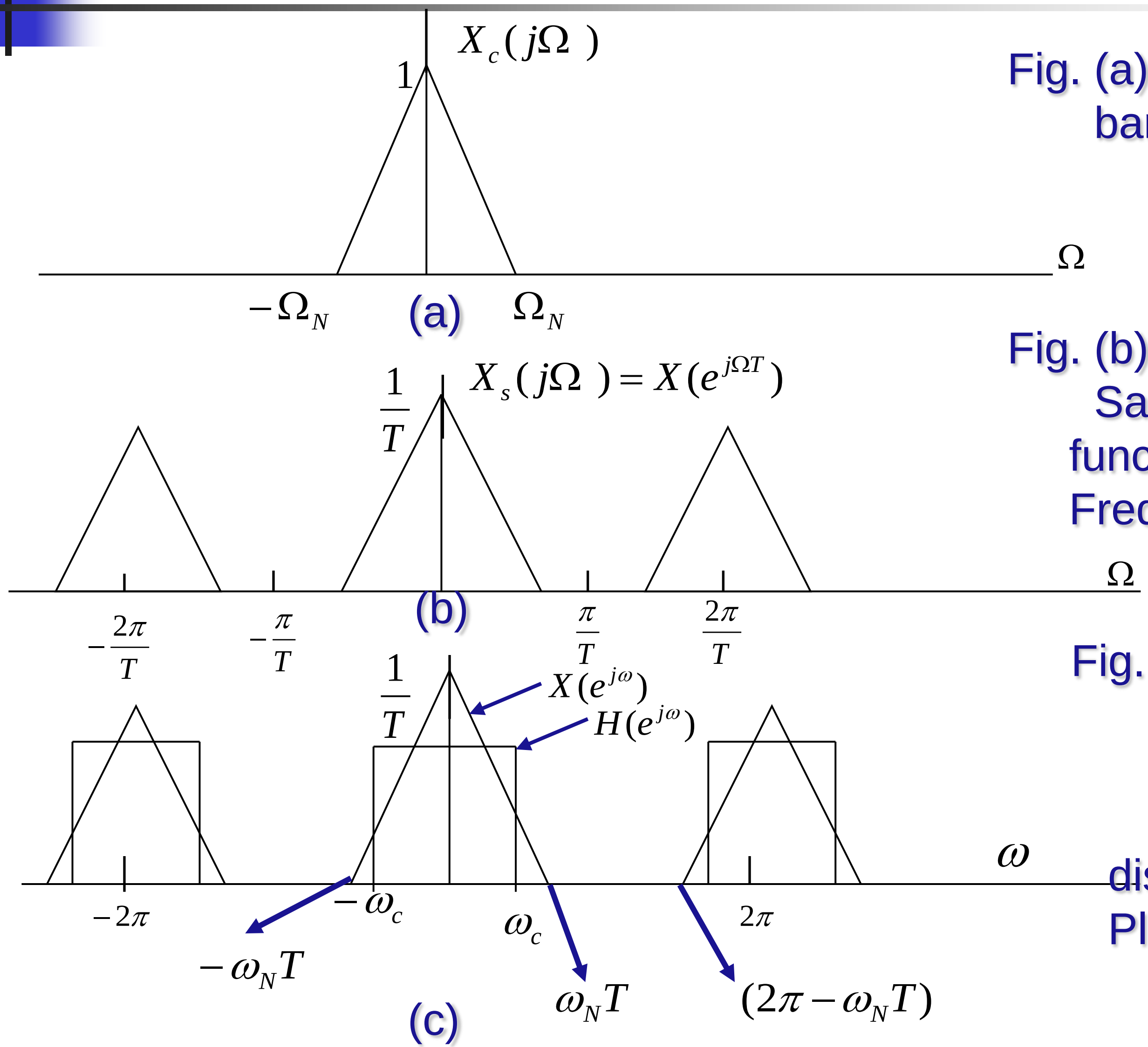


Fig. (a) Fourier transform of band limited input signal

Fig. (b) Fourier transform of Sampled input Plotted as function of continuous-time Frequency Ω

Fig. (c) Fourier transform $X(e^{j\omega})$ of sequence Samples and Frequency Response $H(e^{j\omega})$ of discrete time system Plotted Vs. ω

Changing The Sampling Rate Using Discrete-Time Processing

Sampling Rate Reduction by an Integer Factor

Decimation/Downsampling/Compressor

The process of sampling rate reduction is called Decimation.

The sampling rate of a sequence can be reduced by "sampling it, i.e., by defining a new sequence

$$x_d[n] = x[nM] = x_c(nMT)$$

can be obtained directly from $x_c(t)$ **by sampling with period** $T' = MT$ **.Furthermore, if** $x_c(j\Omega) = 0 \_ \mathit{for} \_ |\Omega| > \Omega_N$ **then** $x_d[n]$

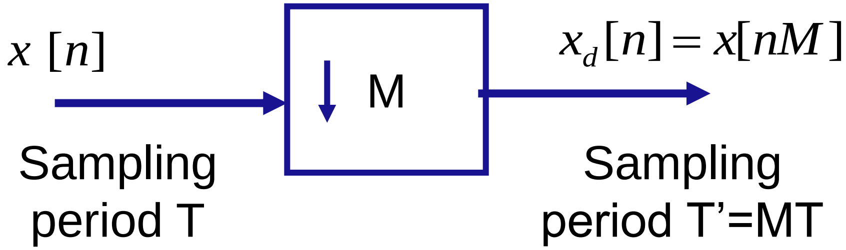
Is an exact representation of $x_c(t)$ **if** $\pi / T' = \pi / (MT) > \Omega_N$
That is, the sampling rate can be reduced by a factor of M with out aliasing If the original sampling rate was at least M times the Nyquist rate.

The operation of reducing the sampling rate is called downsampling or decimation.

Changing The Sampling Rate Using Discrete-Time Processing

Sampling Rate Reduction by an Integer Factor

Representation of Downsampler or discrete-time sampler



We discussed that the discreet-time Furrier Transform of

$x[n] = x_c(nT)$ **is**

$$x\,(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} x_c(j\frac{\omega}{T} - j\frac{2\pi k}{T})$$

Changing The Sampling Rate Using Discrete-Time Processing

Sampling Rate Reduction by an Integer Factor

Similarly, the discreet-time Furrier Transform of

$x_d[n] = x[nM] = x_c(nMT)$ **With** T'=MT is

$$x_d(e^{jw}) = \frac{1}{T'} \sum_{k=-\infty}^{+\infty} x_c(j\frac{\omega}{T'} - j\frac{2\pi k}{T'})$$

Now Since T'=MT , we can write the above Equation as

$$x_d(e^{jw}) = \frac{1}{MT} \sum_{k=-\infty}^{+\infty} x_c(j\frac{\omega}{MT} - j\frac{2\pi r}{T})$$

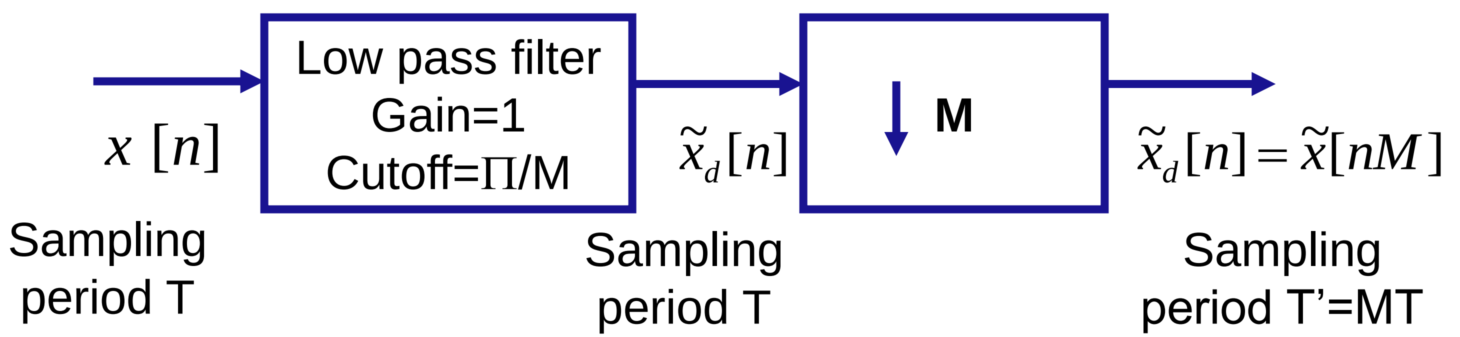
Changing The Sampling Rate Using Discrete-Time Processing

Sampling Rate Reduction by an Integer Factor

Equation for Decimation in Frequency Domain when r in the above equation is expressed as r =i + KM

$$x_d(e^{jw}) = \frac{1}{M}\sum_{i=0}^{M-1} x\left(e^{j(\omega/M - 2\pi i/M)}\right)$$

General System for Sampling rate reduction by M



Changing The Sampling Rate Using Discrete-Time Processing

Frequency-domain illustration of downsampling

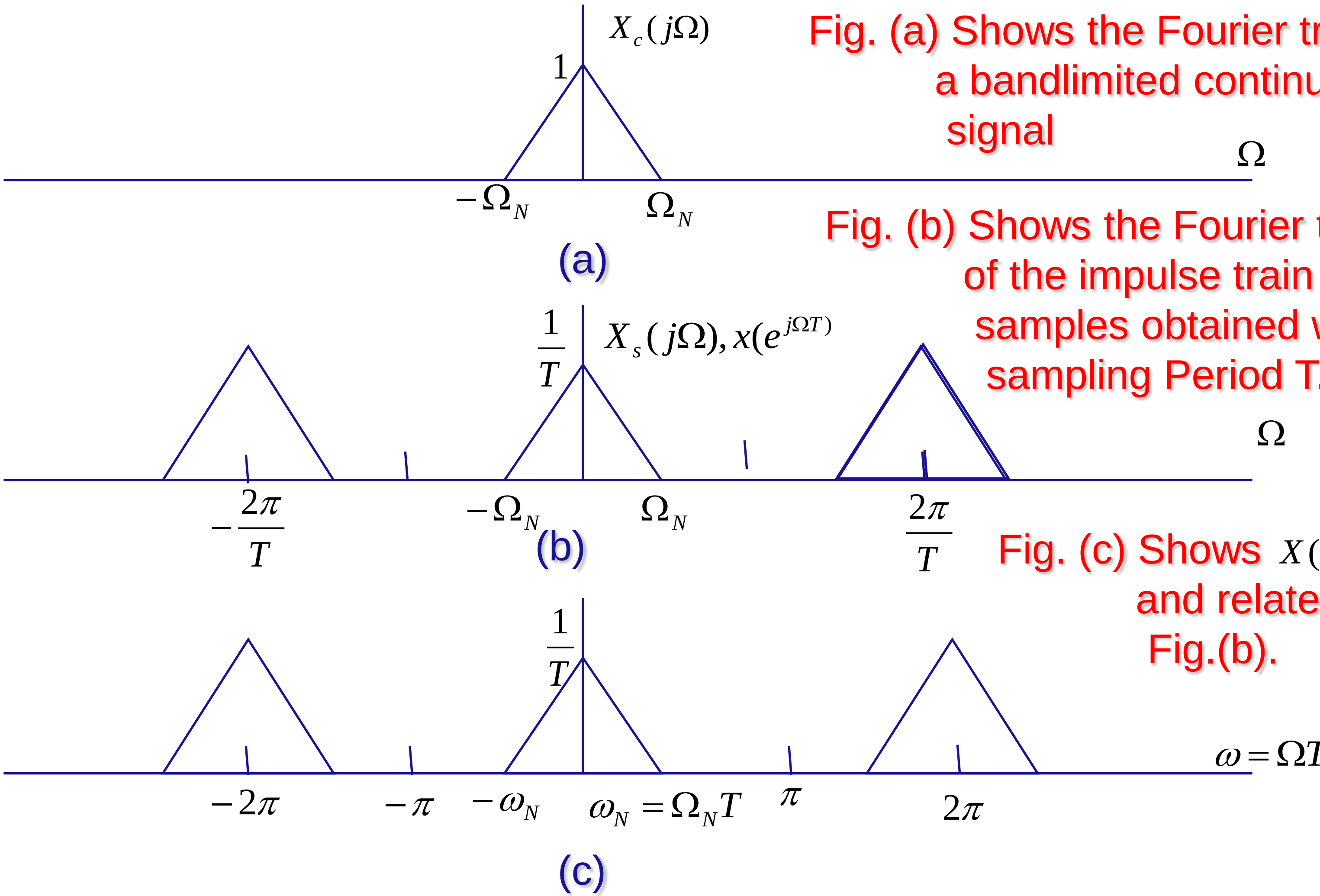


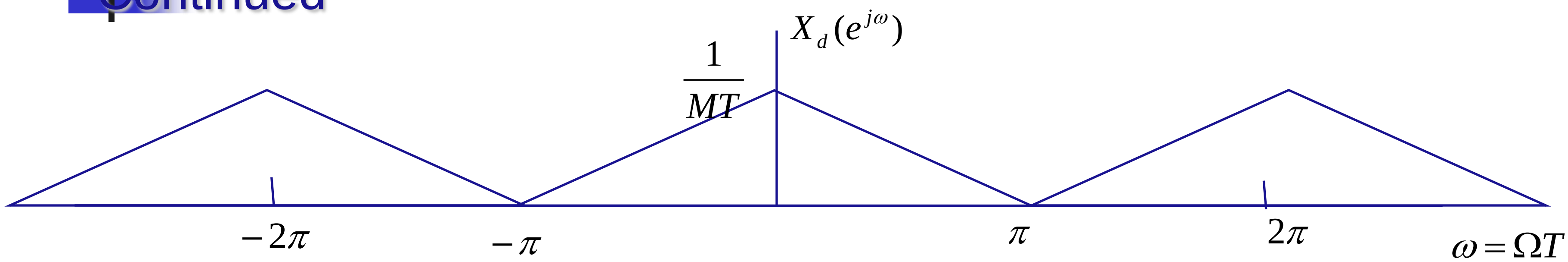
Fig. (a) Shows the Fourier transform a bandlimited continuous time signal

Fig. (b) Shows the Fourier transform of the impulse train of samples obtained with sampling Period T.

Fig. (c) Shows $X(e^{j\omega})$ and related to Fig.(b).

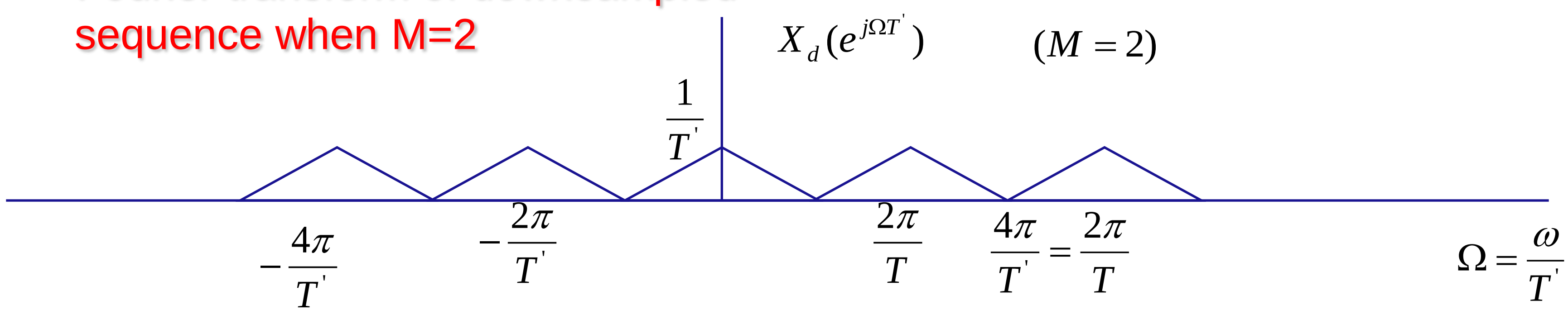
Changing The Sampling Rate Using Discrete-Time Processing

Continued



(D)

Fig. (d) Shows the discrete-time Fourier transform of downsampled sequence when M=2



(e)

Fig. (e) Shows the discrete-time Fourier transform of the downsampled sequence plotted as a function of the continuous time frequency variable Ω

Changing The Sampling Rate Using Discrete-Time Processing

Increasing the Sampling Rate by an Integer Factor

Interpolation

Increasing the sampling rate involves operations analogous to D/C Conversion. To see this consider a signal $x[n]$ whose sampling rate wish to increase by an integer factor of L. If we consider the underlying Continuous-time signal $x_c(t)$, the objective is to obtain samples

$$x_i(n) = x_c(nT')$$

Where $T' = \frac{T}{L}$ **, from the sequence of samples**

$$x[n] = x_c(nT)$$

The operation of increasing the sampling rate is referred as

Upsampling.

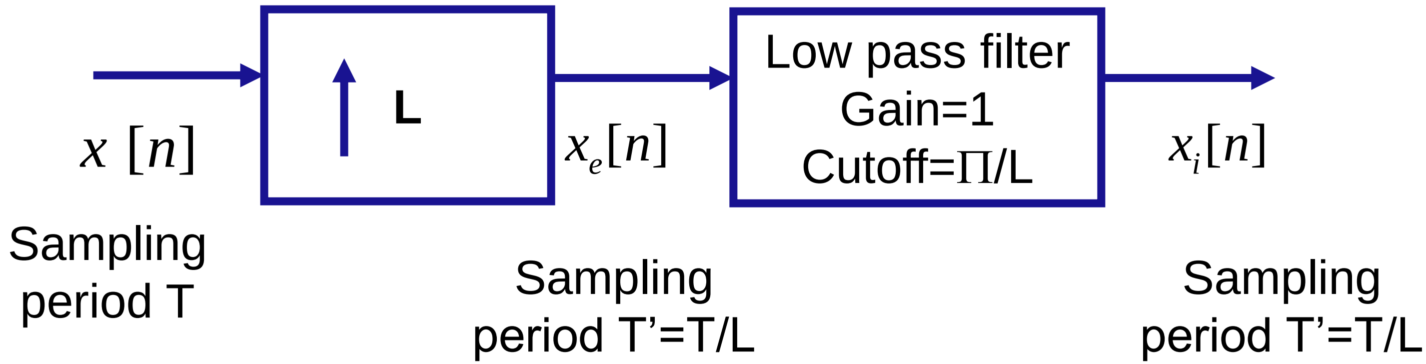
It is clear from the above two equation that

$$x_i[n] = x[n/L] = x_c(nT/L), \qquad n = 0, \pm L, \pm 2L, \ldots.$$

Changing The Sampling Rate Using Discrete-Time Processing

Increasing the Sampling Rate by an Integer Factor

General System for Sampling rate Increased L



The above figure show a system for obtaining $x_i[n]$ from $x[n]$ using only discreet-time processing.

The System on the left is called **Sampling rate Expander** or simply an **expander**. It's out put is

Changing The Sampling Rate Using Discrete-Time Processing

Increasing the Sampling Rate by an Integer Factor

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, ... \\ 0, & otherwise, \end{cases}$$

Or equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - KL]$$

The system on the right (above figure) is a lowpass discrete-time system with cutoff frequency on Or equivalently, Π/L and gain L. This system plays a role similar to the ideal D/C converter.

Changing The Sampling Rate Using Discrete-Time Processing

Increasing the Sampling Rate by an Integer Factor

The Fourier transform of can be expressed as

$$x_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-KL]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\overset{x_e[n]}{\infty}} x[k]e^{-j\omega Lk} = X(e^{j\omega L})$$

Thus the Fourier transform of the out put of the expander Is a frequency scaled version of the Fourier transform of the input, i.e., Ω is replaced by ΩL so that ω is Normalized by $\omega = \Omega T'$

Changing The Sampling Rate Using Discrete-Time Processing

Frequency-domain illustration of interplation

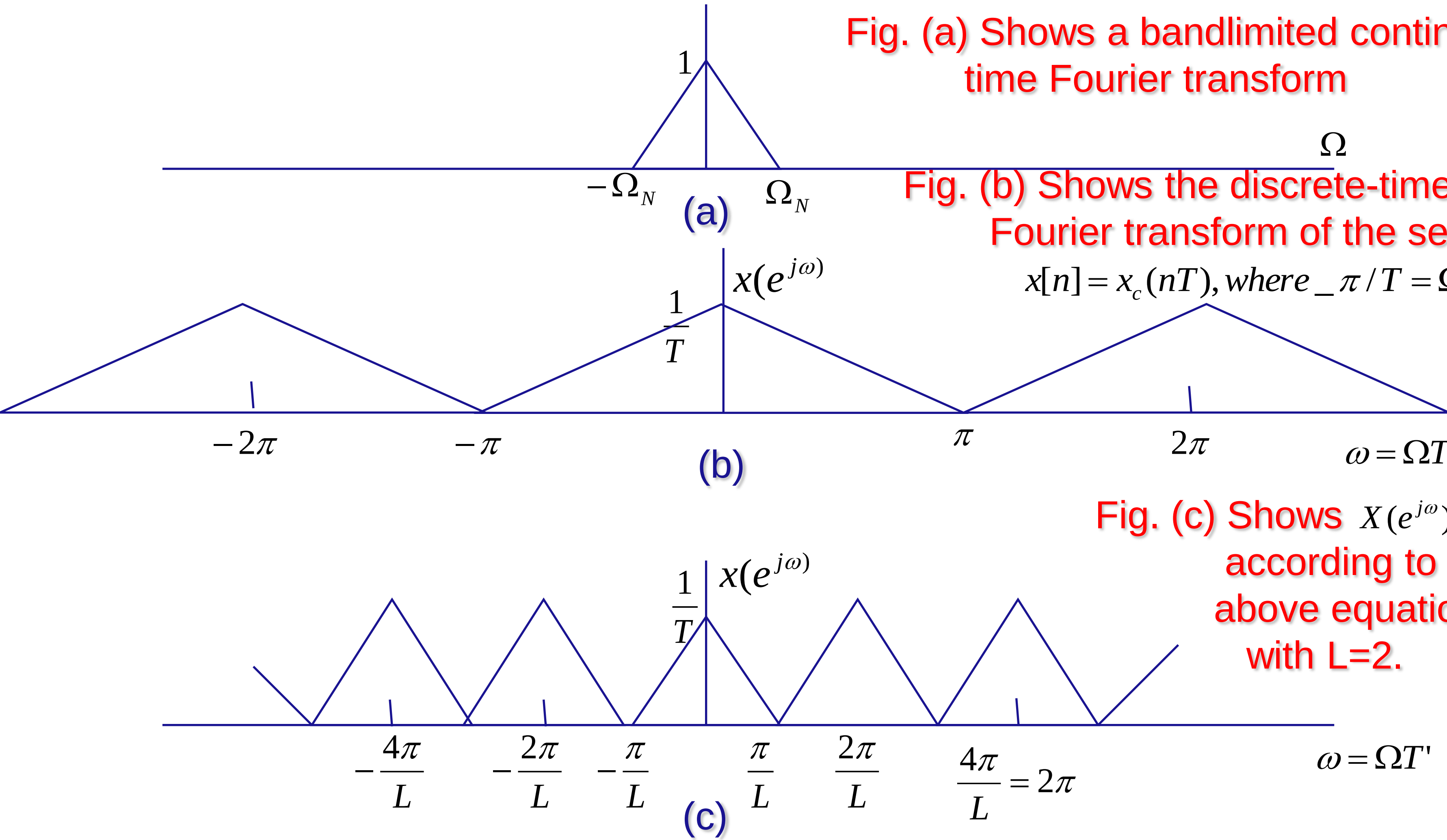


Fig. (a) Shows a bandlimited continuous time Fourier transform

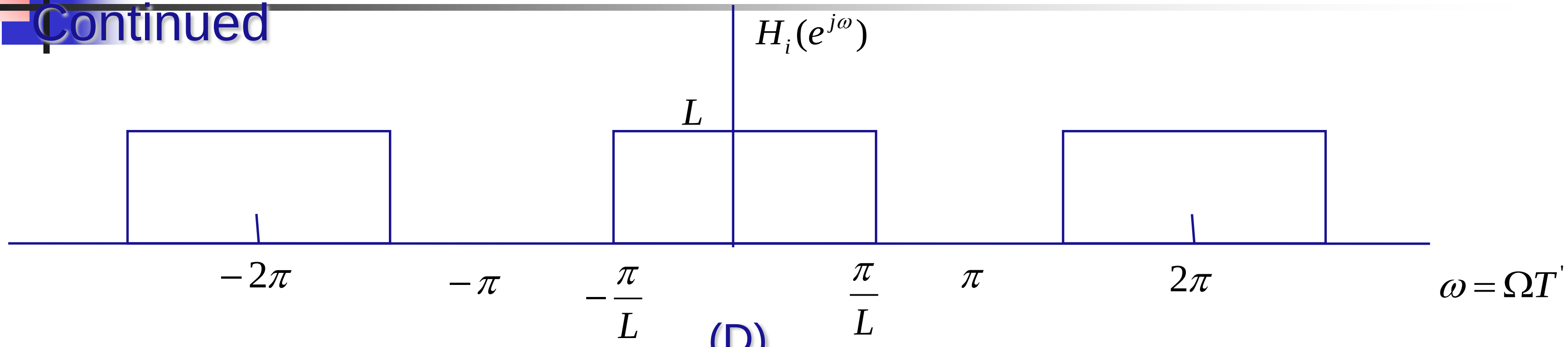
Fig. (b) Shows the discrete-time Fourier transform of the sequenc

$x[n] = x_c(nT), where\_\pi / T = \Omega_N$

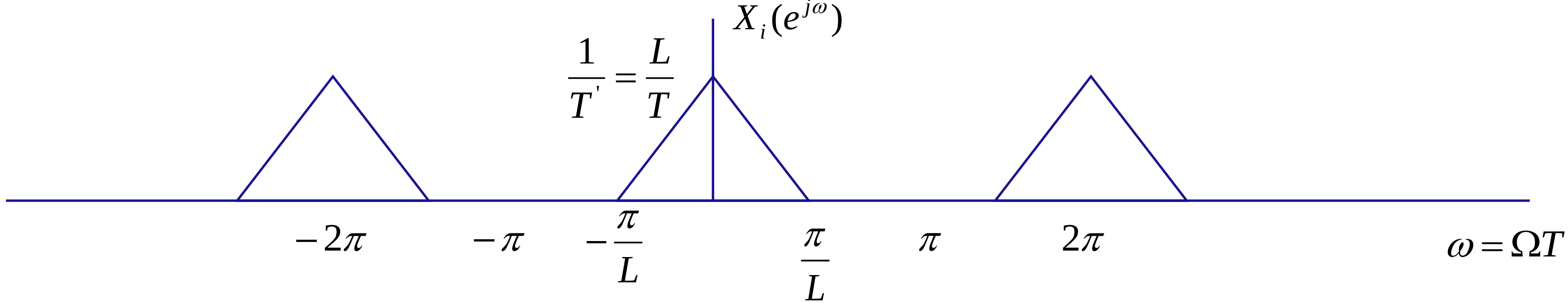
Fig. (c) Shows $X(e^{j\omega})$ according to the above equation, with L=2.

Changing The Sampling Rate Using Discrete-Time Processing

Continued



(D)



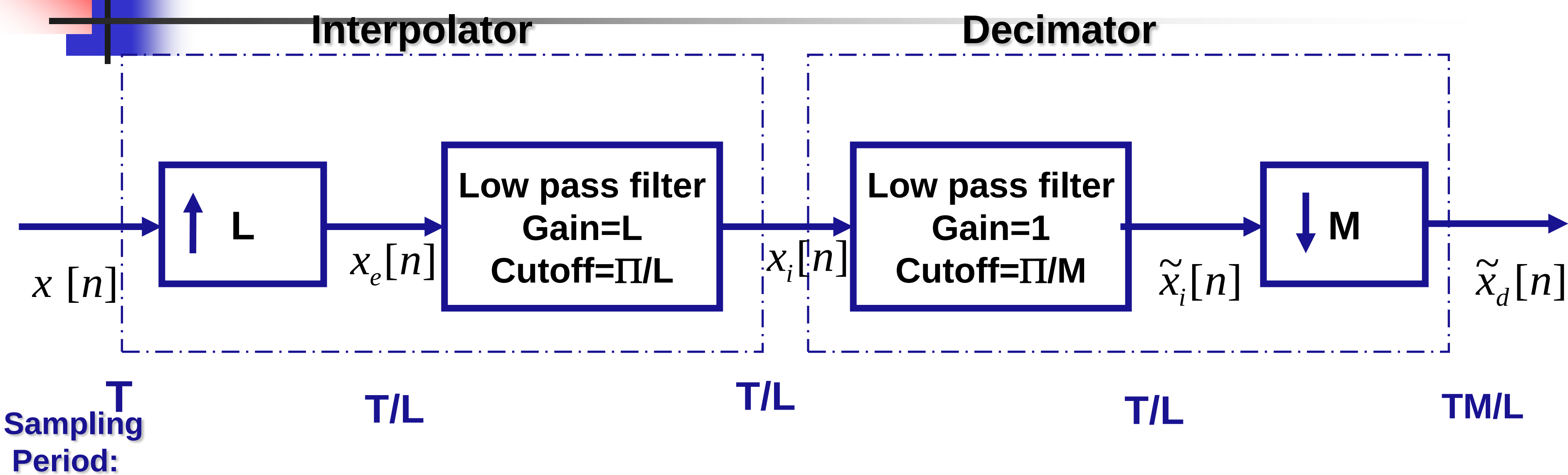
(e)

Fig. (e) Shows the the Fourier transform of the desired signal $x_i[n]$ We see that $X_i(e^{j\omega})$ Can be obtained from $X_e(e^{j\omega})$ by correcting the amplitude scale from 1/T to 1/T' and by removing all the frequency-scaled images of $X_e(e^{j\omega})$ except at integer multiple of 2Π.

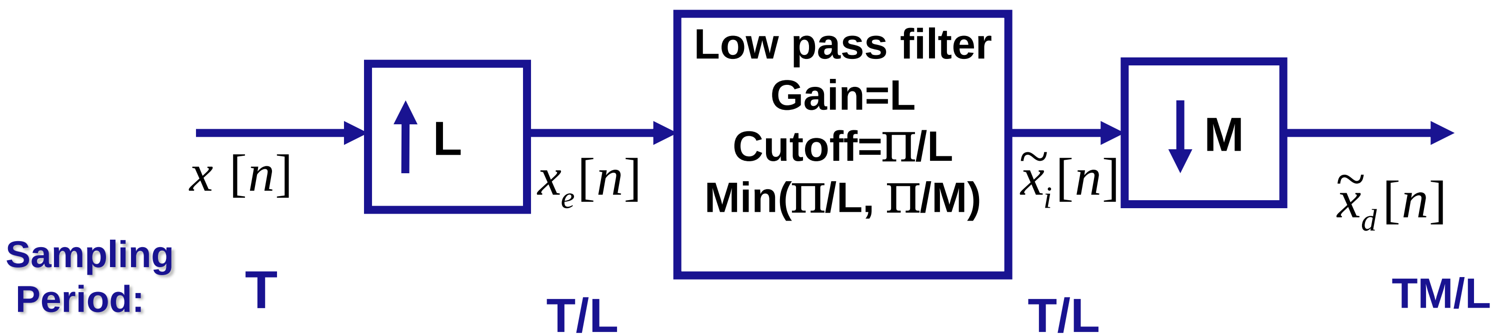
Changing The Sampling Rate By a Noninteger Factor

- By combining decimation and interpolation it is possible to change the sampling rate by noninteger factor.
- An interpolator which decrease the sampling period from T to T/L, Followed by Decimator which increase the sampling period by M, produce an output sequence $\tilde{x}_d[n]$ that has an effective sampling period of T'=TM/L.
  (See the following fig.)

Changing The Sampling Rate By a Noninteger Factor



System for changing the sampling rate by a noninteger factor.



Simplified system in which the decimation and interpolation filters are combined.

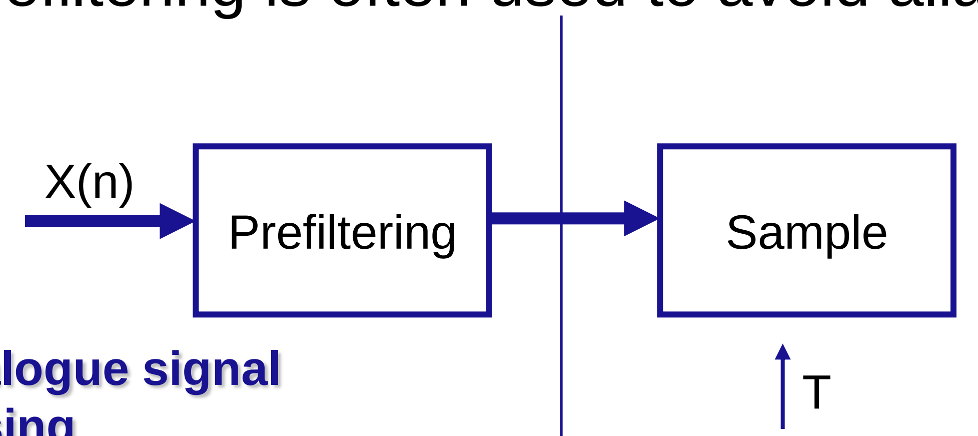
Practical Consideration in AD/DA conversion

Practical Problems:

- Continuous-time signals are not band limited
- Ideal lowpass filter is impossible to be realized

Prefiltering to Avoid Aliasing

When processing analogue system if the input signal is not band limited or if the Nyquist frequency of the input is to high, prefiltering is often used to avoid aliasing.



Prefiltering the analogue signal to reduce anti-aliasing

Continued.

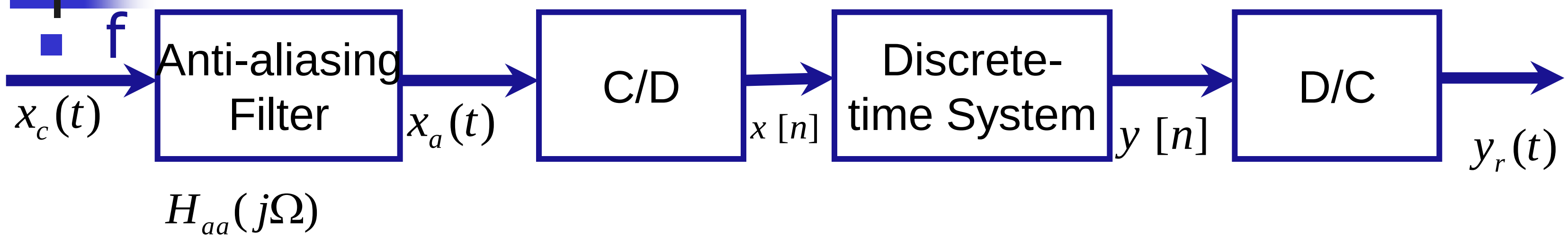


Fig. Use of Prefiltering to avoid aliasing

For an ideal lowpass anti-aliasing filter (above fig.) behaves as a linear time-invariant system with frequency response given by the following equation even When $x_c(j\Omega)$ is not bandlimited.

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega t}) & |\Omega| < \pi / T, \\ 0, & |\Omega| > \pi / T. \end{cases}$$

In practice, the frequency response $H_{aa}(j\Omega)$ can not be ideally bandlimited, but $H_{aa}(j\Omega)$ Can be made small for $|\Omega| < \pi / T$, so that the aliasing is minimized. In this case, the Overall frequency response of the system in the above fig. would be approximately

$$H_{eff}(j\Omega) \approx H_{aa}(j\Omega) H(e^{j\Omega T})$$

Chapter-7
Digital Filter Design

Filter can be defined as a system that modifies certain frequencies relative to others.

Digital filter is a linear shift invariance system (LIS).

The designing filter involves the following stages:

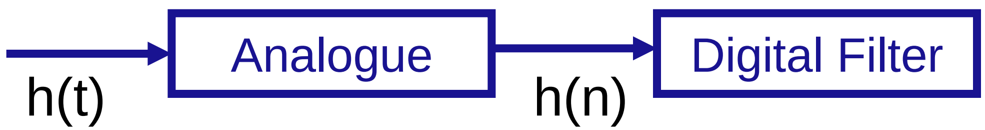
1) Desired characteristics (Specification) of the system.
2) Approximation of the specification using a casual discrete-time system.
3) The realization of the system (building the filter by finite arithmetic computation.

Design of Discrete-Time IIR Filters From Continuous-Time Filters

The traditional approach to the design of discrete-time IIR filters involves the transformation of continuous-time filter into a discrete-time filter meeting prescribed specification.

1. Filter Design by Impulse Invariance

Analogue filter can be changed to digital filter by sampling the impulse response h(t) of analogue. (concept of impulse invariance)



In the impulse invariance design procedure the impulse response of the discrete-time filter is chosen as equally spaced samples of the impulse response of the continuous-time filter; i.e.

$$h[n] = T_d h_c(nT_d)$$ Where: T_d=sampling interval

Note: Impulse invariance techniques have problem of aliasing

Continued..

To develop the transformation (from continuous-time to discrete-time), let us consider the simple function of the continuous time filter expressed in terms of partial fraction expression, as:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

The corresponding impulse response is

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \geq 0, \\ 0, & t < 0. \end{cases}$$

The Impulse response of the discrete-time filter obtained by sampling $T_d h_c(t)$ **is**

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n]$$

$$= \sum_{k=1}^{N} T_d A_k e^{s_k T_d} u[n]$$

Continued..

The system function H (z) of the discrete-time filter is therefore given by

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

2. Bilinear Transformation

- This technique have in distortion of frequency axis.
- Avoid the problem of aliasing.

With $H_c(s)$ denoting the continuous-time system function and H(z) the discrete-time System function, the bilinear transformation corresponds to replacing s by

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right),$$

That is,

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right].$$

FIR Design by Window

IIR filter design are based on transformation of continuous-time IIR system in to Discrete time system.

In contrast, **FIR filters** are almost entirely restricted to discrete-time implementations.

The design technique for FIR filters are based on directly approximating the desired frequency response of the discrete-time system,

The simplest method of FIR filter design is called the Window method.

This method generally begins with an ideal desired frequency response that can be represented as:

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n},$$

Continued…

Where $H_d[n]$ is the corresponding impulse response sequence, which can be expressed in terms of $H_d(e^{jw})$ as

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

To obtain a casual FIR filter from $H_d[n]$ is to define a new system with impulse response h[n] given by

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

Continued…

More generally we can represent h[n] as the product of desired impulse response and a finite-duration "window" w[n]; i.e.'

$$h[n] = h_d[n]w[n],$$

Where for simple truncation as in above equation the window is the rectangular window

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

It follows from the modulation or windowing theorem that

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Properties of commonly used windows

Some commonly used windows are defined by the following equations:

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2 \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & otherwise \end{cases}$$

Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

The Kaiser Window Filter Design Method

The Kaiser window is defined as

$$w[n] = \begin{cases} \dfrac{I_o[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_o(\beta)} & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

Where α=M/2, and $I_o(\cdot)$ represents the zero-order modified Bessel function of the first kind.

Properties of Linear phase FIR Filter

The Shape of the impulse response defined by Equation.

$$H(z) = \sum_{n=0}^{M-1} h(n) = Z^{-n} = Z^{-(m-1)} \sum_{n=0}^{M-1} h(n) Z^{M-1-n}$$

The frequency response of the above system is

$$H(e^{jw}) = \sum_{n=0}^{M-1} h(n) e^{j\omega n} \quad -\pi \le \omega \le \pi$$

Taking $\angle H(e^{j\omega}) = -\alpha\omega$ where $\alpha = \frac{M-1}{2}$ in linear phase and with symmetry condition $h(n) = h(M-1-n)$

1) Case 1 when M is odd

$$\alpha = \frac{M-1}{2}$$ integer

Continued..

2) Case 2 when M is even

$$\alpha = \frac{M-1}{2}$$ **Non integer**

For other linear phase $\angle H(e^{j\omega}) = \beta - \alpha\omega$ and anti symmetric condition $h(n) = -h(m-1-n)$ which is opposite of symmetry.

1) Case 1 when M is odd

$$\alpha = \frac{M-1}{2}$$ **integer**

2) Case 2 when M is even

$$\alpha = \frac{M-1}{2}$$ **Non integer**

Of FIR Filters With Generalized Linear Phase

Type I

In designing a causal Type I linear phase FIR filter, it is convenient first to consider the design of a zero-phase filter, i.e., one for which

$$h_e[n] = h_e[-n]$$

and then to insert sufficient delay to make it causal.

Type I For Linear phase and symmetry

M: odd, β=0, $\alpha = \dfrac{M-1}{2}$

$$h(n) = h(M-1-n)$$ Symmetry condition

$$H(e^{j\omega}) = \left[\sum_{n=0}^{M-1/2} a(n)\cos w.n\right] e^{-j\omega\frac{M-1}{2}}$$

$$a(0) = h\left(\frac{M-1}{2}\right)$$ Is the middle sample

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Type II

A Type II causal filter is one for which h[n]=0 outside the range $0 \le n \le M$, with filter length (M+1) even, I.e.

M: even, and with the symmetry property

$$h(n) = h(M-1-n)$$

$\beta = 0$ and $\alpha = \dfrac{M-1}{2}$ Not integer

The frequency response $H(e^{j\omega})$ can be expressed in the form

$$H(e^{j\omega}) = \left\{ \sum_{n=1}^{M/2} b(n) \cos(w(n-1/2)) \right\} e^{-j\omega\frac{M-1}{2}}$$

When $b(n) = 2h(\frac{M}{2} - n), n = 1,2,\ldots(M+1/2),$

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Type III

For Linear Anti-symmetric

When $\beta = \pi / 2$ and $\alpha = \frac{M-1}{2}$ integer

$$h(n) = -h(M-1-n)$$ Anti-symmetric

The frequency response $H(e^{j\omega})$ can be expressed in the form

$$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{M-1}{2}} c(n) \sin(\omega n) \right] e^{j\left[\frac{\pi}{2} - (\frac{M-1}{2})\omega\right]}$$

Where $c(n) = 2h(\frac{M-1}{2})$

at $H_r(\omega) = 0\_ for\_\omega = 0\_\omega = \pi$

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Type IV

For Linear Anti-symmetric

When $\beta = \pi / 2$ and $\alpha = \dfrac{M-1}{2}$ not integer

$$h(n) = -h(M-1-n) \quad \text{Anti-symmetric}$$

The frequency response $H(e^{j\omega})$ can be expressed in the form

$$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{M}{2}} d(n) \sin[\omega(n - \frac{1}{2})] \right] e^{j\left[\frac{\pi}{2} - \omega(\frac{M-1}{2})\right]}$$

Where $d(n) = 2h(\dfrac{M}{2} - n)$

at $\omega = 0, H_r(0) = 0 \qquad \angle H(0) = e^{j.\pi/2} = j$

Thank you all